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Multidimensional Lindstedt–Poincaré method for nonlinear vibration of axially moving beams

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Abstract

The multidimensional Lindstedt–Poincaré (MDLP) method is extended to the nonlinear vibration analysis of axially moving systems. Galerkin method is used to discretize the governing equations. The forced response of an axially moving beam with internal resonance between the first two transverse modes is studied. The fundamental harmonic resonance is studied. The response curves exhibit the same internal resonance characteristics as that of non-transferring thin plates and beams because all these systems possess cubic nonlinearity and similar frequency distribution. The examples show that the results of the MDLP method agree reasonably well with that obtained by the incremental harmonic balance (IHB) method. However, the former is more straightforward and efficient for obtaining the solution.

1. Introduction

Magnetic tapes, power transmission belts, band saws, aerial cable tramways and pipes conveying fluid can be considered as axially moving systems. Investigations have been conducted on the linear and nonlinear responses of axially moving strings and beams. Among them, Ulsoy et al. [1] and Wickert and Mote [2] presented comprehensive reviews on the subject area up to 1978 and 1988, respectively. More recently, Wickert [3] analyzed the nonlinear vibration and bifurcation of axially moving beams through the Krylov–Bogoliubov–Mitropolsky asymptotic method. Pellicano and Vestroni [4,5] studied the bifurcation, the post-bifurcations velocity with viscous damping and external harmonic excitation. Pellicano et al. [6] also studied the stability of parametrically excited axially moving systems by experimental and theoretical means. Chen and his co-workers [7–9] investigated the bifurcations and chaos of an axially moving viscoelastic and geometric nonlinear string/beam. They studied the nonlinear dynamics behavior of the traveling system with time-dependent axial velocity and tension. Zhang and Zu [10,11] used the method of multiple scales to study the dynamic response and stability parametrically for viscoelastic and geometric nonlinear moving belts. Fung and Chang [12] employed the finite difference method with variable grid for numerical computation of the string/slider nonlinear coupling system with time-dependent boundary condition. Öz et al. [13,14] also applied

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the method of multiple scales to study the nonlinear vibrations and stability of axially moving beams and tensioned pipes conveying fluid at harmonically varying speeds. Suweken and Horssen [15–17] used a two time-scale perturbation method to approximate the solutions of a conveyor belt moving at low and time-varying velocities. Cao and Zhang [18] obtained the quasi-periodic solutions of the coupled string-beam system subjected to a harmonic axial load by the method of multiple scales.

Complex dynamic behavior of the axially moving system occurs when the excitation frequency Ω is the sum, difference and small multiples of its natural frequencies. Pellicano et al. [19] considered the primary resonance and the parametric resonance that occur when Ω is near the first natural frequency ω_1 and $2\omega_1$, respectively. Chen et al. [20] studied the dynamic stability of an axially accelerating beam. Sub-harmonic and combination resonances were considered. On the contrary, only a few studies have been devoted to the internal resonance of the axially moving system [21]. In this aspect, Riedel and Tan [22] studied the forced responses of an axially moving strip with internal resonance when Ω is near ω_1. The method of multiple scales is used to conduct the perturbation analysis and to determine the frequency response numerically at low and high axial velocities. Suweken and Van Horssen [17] investigated the complicated dynamical behavior for sum-type and difference-type of internal resonances on the transverse vibrations of a conveyor belt with time-varying velocity. The stability properties of the belt system were demonstrated. The present authors [23] have studied the forced response of an axially moving strip with internal resonance by using the incremental harmonic balance (IHB) method developed by Lau et al. [24–26].

Note worthily, Lau et al. [27] and Chen et al. [28] developed an alternative perturbation procedure of multiple scales for the nonlinear vibration analysis of multi-degree-of-freedom systems. In this paper, the method is extended to the analysis of nonlinear vibration of axially moving beams which belong to the gyroscopic system. The method can be considered as a generalization of the Lindstedt–Poincaré method to multidimensional systems and will be termed as the multidimensional Lindstedt–Poincaré (MDLP) method. This paper starts with a brief description on the governing equation of the axially moving system followed by an introduction on the essence of the MDLP method. Typical cases of the axially moving beam problem will be investigated. Results will be presented and compared with that obtained by the IHB method.

2. Governing equation for axially moving beam

A beam passing through two simple supports at constant axial or transport velocity V is considered. Properties of the beam include its cross-sectional area A, mass density ρ and flexural rigidity EI. The beam is tensioned by a force P and oscillates in the X–Z-plane with the transverse displacement $W(X, T)$ where T denotes time, see the sketch in Fig. 1. From previous studies, the natural frequencies of the transverse vibration are much larger than that of the longitudinal vibration [23]. Their coupled effect is weak and we will focus on the forced transverse vibrations with the longitudinal ones neglected. The material transverse velocity of the beam is

$$\frac{\mathrm{d}W}{\mathrm{d}T} = \frac{\partial W}{\partial T} + \frac{\partial W}{\partial X}\frac{\partial X}{\partial T} = W_{,T} + VW_{,X}. \tag{1}$$

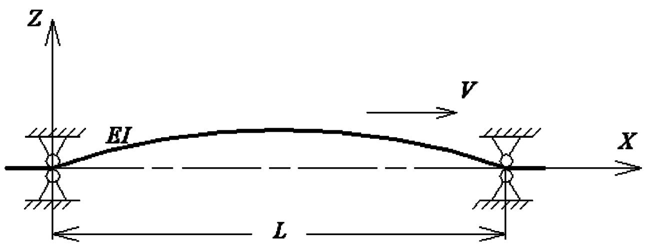

Fig. 1. Schematic diagram for an axially moving beam.

The pertinent kinetic energy T_k and potential energy U_p of the beam are

$$T_k = \frac{\rho A}{2}\int_0^L (W_{,T} + VW_{,X})^2\,\mathrm{d}X, \quad U_p = \int_0^L \left(P\varepsilon_X + \frac{1}{2}\,\mathrm{EA}\,\varepsilon_X^2 + \frac{1}{2}\,\mathrm{EI}\,W_{,XX}^2\right)\mathrm{d}X, \tag{2}$$

where the von Karman approximation is employed for the nonlinear axial strain, i.e., $\varepsilon_x = (W_{,x})^2/2$. The Hamilton Principle states

$$\delta\int_{T_1}^{T_2}(T_k - U_p)\,\mathrm{d}T = 0. \tag{3}$$

By virtue of Eq. (2), the Principle can be expressed as

$$\delta\int_{T_1}^{T_2}\int_0^L F(T, X, W_{,T}, W_{,X}, W_{,XX})\,\mathrm{d}X\,\mathrm{d}T = 0, \tag{4}$$

where

$$F = \frac{\rho A}{2}(W_{,T} + VW_{,X})^2 - \left[P\varepsilon_X + \frac{1}{2}\,\mathrm{EA}\,\varepsilon_X^2 + \frac{1}{2}\,\mathrm{EI}\,W_{,XX}^2\right].$$

Further manipulation yields:

$$\int_{T_1}^{T_2}\int_0^L\left[-\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial W_{,X}}\right) - \frac{\partial}{\partial T}\left(\frac{\partial F}{\partial W_{,T}}\right) + \frac{\partial^2}{\partial X^2}\left(\frac{\partial F}{\partial W_{,XX}}\right)\right]\delta W\,\mathrm{d}X\,\mathrm{d}T = 0. \tag{5}$$

Hence, the Euler equation is

$$\frac{\partial}{\partial T}\left(\frac{\partial F}{\partial W_{,T}}\right) + \frac{\partial}{\partial X}\left(\frac{\partial F}{\partial W_{,X}}\right) - \frac{\partial^2}{\partial X^2}\left(\frac{\partial F}{\partial W_{,XX}}\right) = 0. \tag{6}$$

After some differential operations, the governing equation can be derived to be

$$\rho A W_{,TT} + 2\rho A V W_{,XT} + \rho A V^2 W_{,XX} - (P + \tfrac{3}{2}\mathrm{EA}\,W_{,X}^2)W_{,XX} + \mathrm{EI}\,W_{,XXXX} = 0. \tag{7}$$

For simplicity, the following parameters are introduced:

$$(w, x, t, v, v_1, v_f) = (W/L, X/L, T\sqrt{P/\rho A L^2}, V/\sqrt{P/\rho A}, \sqrt{EA/P}, \sqrt{EI/PL^2}), \tag{8}$$

where $w(x,\ t)$, v, v_1 and v_f are respectively the dimensionless lateral deflection, constant transport velocity, longitudinal stiffness parameter and flexural stiffness parameter, respectively. With these parameters, Eq. (7) can be expressed non-dimensionally as

$$w_{,tt} + 2vw_{,xt} + (v^2 - 1)w_{,xx} - \tfrac{3}{2}v_1^2 w_{,x}^2 w_{,xx} + v_f^2 w_{,xxxx} = 0, \tag{9}$$

where $w_{,tt}$, $2vw_{,xt}$ and $v^2w_{,xx}$ are respectively the local, Coriolis and centripetal accelerations. The supporting conditions are

$$w(0,t) = w(1,t) = 0, \quad w_{,xx}(0,t) = w_{,xx}(1,t) = 0. \tag{10}$$

The following separable solution in terms of admissible functions can be assumed as

$$w(x,t) = \sum_{j=1}^{N} q_j(t)\sin(j\pi x). \tag{11}$$

By substituting Eq. (11) into Eq. (9), multiplying all the terms with $\sin(j\pi x)$ and integrating the resulting equation from $x = 0$ to $x = 1$, the following second-order ordinary differential equations can be obtained as

$$\sum_{j=1}^{N} M_{ij}\ddot{q}_j + \sum_{j=1}^{N} G_{ij}\dot{q}_j + \sum_{j=1}^{N} K_{ij}q_j + \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} K_{ijkl}q_l q_k q_j = 0, \quad i = 1, 2, \ldots, N, \tag{12}$$

where $\dot{q}_j = \mathrm{d}q_j/\mathrm{d}t$ and $\ddot{q}_j = \mathrm{d}^2q_j/\mathrm{d}t^2$. Moreover, M_{ij}, G_{ij}, K_{ij} and K_{ijkl} are respectively the mass, gyroscopic, linear stiffness and cubic stiffness coefficients. In particular,

$$M_{ij} = \int_0^1 \sin i\pi x \sin j\pi x \, \mathrm{d}x = \tfrac{1}{2}\delta_{ij},$$

$$K_{ij} = \left[v_f^2 j^4\pi^4 - (v^2-1)j^2\pi^2\right]\int_0^1 \sin i\pi x \sin j\pi x \, \mathrm{d}x,$$

$$G_{ij} = 2vj\pi \int_0^1 \sin i\pi x \cos j\pi x \, \mathrm{d}x = \begin{cases} 4ijv/(i^2-j^2) & \text{for} \quad i \neq j \text{ and even } i+j, \\ 0 & \text{otherwise,} \end{cases}$$

$$K_{ijkl} = \tfrac{3}{2}v_1^2 j^2 kl\pi^4 \int_0^1 \sin i\pi x \sin j\pi x \cos k\pi x \cos l\pi x \, \mathrm{d}x = \tfrac{3}{2}v_1^2 j^2 kl\pi^4 I_{sscc}(i,j,k,l),$$

$$I_{sscc}(i,j,k,l) = \tfrac{1}{4}[I_{cc}(i-j,\, k+l), + I_{cc}(i-j,\, k-l) - I_{cc}(i+j, k+l) - I_{cc}(i+j, k-l)],$$

$$I_{cc}(i,j) = \int_0^1 \cos i\pi x \cos j\pi x \, \mathrm{d}x = \begin{cases} 0 & \text{for} \quad i \neq j, \\ 1/2 & \text{for} \quad i = j \neq 0, \\ 1 & \text{for} \quad i = j = 0. \end{cases}$$

If two transverse modes are considered, we let $N = 2$ and Eq. (12) would yield:

$$\ddot{q}_1 - \mu_{12}\dot{q}_2 + k_{11}q_1 + \bar{k}_{12}q_1q_2^2 + \bar{k}_{13}q_1^3 = 0, \quad \ddot{q}_2 + \mu_{21}\dot{q}_1 + k_{21}q_2 + \bar{k}_{22}q_2q_1^2 + \bar{k}_{23}q_2^3 = 0, \tag{13}$$

where

$$\mu_{12} = \mu_{21} = 16v/3, \quad k_{11} = (v_f^2\pi^2 - v^2 + 1)\pi^2, \quad \bar{k}_{12} = 3v_1^2\pi^4, \quad \bar{k}_{13} = 3v_1^2\pi^4/8,$$
$$k_{21} = 4(4v_f^2\pi^2 - v^2 + 1)\pi^2, \quad \bar{k}_{22} = 3v_1^2\pi^4, \quad \bar{k}_{23} = 6v_1^2\pi^4.$$

The associated autonomous linear conservation system is governed by the following gyroscopic equations:

$$\ddot{q}_1 - \mu_{12}\dot{q}_2 + k_{11}q_1 = 0, \quad \ddot{q}_2 + \mu_{21}\dot{q}_1 + k_{21}q_2 = 0. \tag{14}$$

On the other hand, the natural frequencies ω_{10} and ω_{20} can be solved from

$$\omega^4 - (k_{11} + k_{21} + \mu_{12}\mu_{21})\omega^2 + k_{11}k_{21} = 0. \tag{15}$$

3. Multidimensional Lindstedt–Poincaré method

Lau et al. [27] and Chen et al. [28] developed an alternative perturbation procedure of multiple scales for nonlinear analysis of multi-degree-of-freedom vibrating systems. In this section, the method is extended to nonlinear vibration analysis of axially moving beams which belong to the gyroscopic system. The method is indeed a generalization of the Lindstedt–Poincaré method to multidimensional systems and it will be termed as the MDLP method in this paper.

For the forced response of the system under consideration, modal damping terms ($\bar{\mu}_{11}$ and $\bar{\mu}_{22}$) and excitation terms ($\bar{f}_{11}$ and $\bar{f}_{21}$) can be incorporated into Eq. (13), i.e.

$$\ddot{q}_1 + \bar{\mu}_{11}\dot{q}_1 - \mu_{12}\dot{q}_2 + k_{11}q_1 + \bar{k}_{12}q_1q_2^2 + \bar{k}_{13}q_1^3 = \bar{f}_{11}\cos \Omega t, \tag{16}$$

$$\ddot{q}_2 + \mu_{21}\dot{q}_1 + \bar{\mu}_{22}\dot{q}_2 + k_{21}q_2 + \bar{k}_{22}q_2q_1^2 + \bar{k}_{23}q_2^3 = \bar{f}_{21}\cos \Omega t. \tag{17}$$

It should be remarked that μ_{12} and μ_{21} are the gyroscopic coefficients which provide an internal damping effect to the system whilst $\bar{\mu}_{11}$ and $\bar{\mu}_{22}$ arise from the external viscous damping.

For the perturbation procedure, a small parameter $\varepsilon \ll 1$ is introduced to the last two equations which would become

$$\ddot{q}_1 - \mu_{12}\dot{q}_2 + k_{11}q_1 + \varepsilon\mu_{11}\dot{q}_1 + \varepsilon k_{12}q_1q_2^2 + \varepsilon k_{13}q_1^3 = \varepsilon f_{11}\cos T, \tag{18}$$

$$\ddot{q}_2 + \mu_{21}\dot{q}_2 + k_{21}q_2 + \varepsilon\mu_{22}\dot{q}_2 + \varepsilon k_{22}q_2q_1^2 + \varepsilon k_{23}q_2^3 = \varepsilon f_{21} \cos T, \tag{19}$$

where

$$\varepsilon k_{ij} = \bar{k}_{ij}, \quad \mu_{ii} = \varepsilon\bar{\mu}_{ii}, \quad \varepsilon f_{i1} = \bar{f}_{i1} \quad \text{for} \quad i = 1,2 \text{ and } j = 2,3 \tag{20}$$

and $T = \Omega t$. Other time variables are introduced as

$$\tau_n = \omega_n t \; (n = 1,2) \tag{21}$$

in which ω_n ($n = 1,2$) are the nonlinear frequency of the response to be sought and variables q_n ($n = 1,2$) can be regarded as functions of the independent variables τ_n. Let q_n and ω_n be expanded in power series of ε, i.e.

$$q_n = \sum_{k=0} q_{nk}\varepsilon^k, \quad \omega_n = \sum_{k=0} \omega_{nk}\varepsilon^k, \tag{22}$$

where ω_{nk} ($k = 1,2,\ldots$) and q_{nk} are unknowns to be determined. Thus, the derivatives of q_n with respect to t can be expressed as

$$\frac{\mathrm{d}q_n}{\mathrm{d}t} = \sum_{i=1} \omega_i \frac{\partial q_n}{\partial \tau_i} = \sum_{k=0}\sum_{l=0} \varepsilon^{k+l} D_k q_{nl}, \quad \frac{\mathrm{d}^2 q_n}{\mathrm{d}t^2} = \sum_{i=1}\sum_{j=1} \omega_i\omega_j \frac{\partial^2 q_n}{\partial\tau_i\partial\tau_j} = \sum_{k=0}\sum_{l=0}\sum_{m=0} \varepsilon^{k+l+m} D_{kl}^2 q_{nm}, \tag{23}$$

where

$$D_k = \sum_{i=1} \omega_{ik}\frac{\partial}{\partial\tau_i}, \quad D_{kl}^2 = D_k D_l = \sum_{i=1}\sum_{j=1} \omega_{ik}\omega_{jl}\frac{\partial^2}{\partial\tau_i\partial\tau_j}.$$

After substituting Eqs. (22) and (23) into Eqs. (18) and (19), equating the coefficients of ε^0 and ε^1 yields the relations below.

Order ε^0:

$$D_{00}^2 q_{10} + k_{11}q_{10} - \mu_{12}D_0 q_{20} = 0, \quad D_{00}^2 q_{20} + k_{21}q_{20} + \mu_{21}D_0 q_{10} = 0. \tag{24}$$

Order ε^1:

$$D_{00}^2 q_{11} + k_{11}q_{11} - \mu_{12}D_0 q_{21} = -\mu_{11}D_0 q_{10} + \mu_{12}D_1 q_{20} - 2D_{01}^2 q_{10} - k_{12}q_{10}q_{20}^2 - k_{13}q_{10}^3 + f_{11} \cos \Omega t, \tag{25}$$

$$D_{00}^2 q_{21} + k_{21}q_{21} + \mu_{21}D_0 q_{11} = -\mu_{22}D_0 q_{20} - \mu_{21}D_1 q_{10} - 2D_{01}^2 q_{20} - k_{22}q_{10}^2 q_{20} - k_{23}q_{20}^3 + f_{21} \cos \Omega t. \tag{26}$$

To show the essential features of the MDLP method, the fundamental harmonic resonance will be studied. For reference purpose, predictions of the IHB method have also been computed by the procedure in Ref. [23]. As a generalization of Lindstedt–Poincaré method from single degree-of-freedom systems to multiple degree-of-freedom systems, it will be seen that the MDLP procedure is inheritably straightforward and convenient for studying the steady-state responses including fundamental resonance, super-harmonic and sub-harmonic resonance under the influence of internal or combination resonances.

With the forcing frequency Ω close to the first natural frequency ω_{10}, fundamental resonance will occur. In this light, one can take $f_{21} = 0$ in Eq. (17) and $\Omega = \omega_1$, i.e.

$$T = \tau_1. \tag{27}$$

When the second natural frequency ω_{20} is nearly three times of the first natural frequency ω_{10}, the internal resonance is likely to occur. By letting $\omega_2 = 3\omega_1$, i.e.

$$\tau_2 = 3\tau_1, \tag{28}$$

the solutions to Eq. (24) can be solved to be

$$q_{10} = a_{10} \cos(\tau_1 + \varphi_1) + a_{20} \cos(\tau_2 + \varphi_2), \quad q_{20} = p_1 a_{10} \sin(\tau_1 + \varphi_1) + p_2 a_{20} \sin(\tau_2 + \varphi_2) \tag{29}$$

in which φ_1 and φ_2 are phase angles and

$$p_1 = \frac{-\omega_{10}^2 + k_{11}}{\mu_{21}\omega_{10}} = \frac{\mu_{12}\omega_{10}}{-\omega_{10}^2 + k_{21}}, \quad p_2 = \frac{-\omega_{20}^2 + k_{11}}{\mu_{12}\omega_{20}} = \frac{\mu_{21}\omega_{20}}{-\omega_{20}^2 + k_{21}}. \tag{30}$$

In the gyroscopic system, q_{10} and q_{20} are not independent of each other. Substitution of Eq. (29) into Eqs. (25) and (26) gives

$$\begin{aligned} D_{00}^2 q_{11} + k_{11}q_{11} - \mu_{12}D_0 q_{21} = {} & R_{11}\cos(\tau_1+\varphi_1) + R_{12}\cos(\tau_2+\varphi_2) \\ & + R_{13}\sin(\tau_1+\varphi_1) + R_{14}\sin(\tau_2+\varphi_2) + \text{NST}, \end{aligned} \tag{31}$$

$$\begin{aligned} D_{00}^2 q_{21} + k_{21}q_{21} + \mu_{21}D_0 q_{11} = {} & R_{21}\sin(\tau_1+\varphi_1) + R_{22}\sin(\tau_2+\varphi_2) \\ & + R_{23}\cos(\tau_1+\varphi_1) + R_{24}\cos(\tau_2+\varphi_2) + \text{NST}, \end{aligned} \tag{32}$$

where NST denotes "non-secular terms". Moreover,

$$\begin{aligned}
R_{11} = {} & 2\omega_{10}\omega_{11}a_{10} + \mu_{12}p_1\omega_{11}a_{10} + \tfrac{1}{4}(-3k_{13}a_{10}^3 - 6k_{13}a_{10}a_{20}^2 - k_{12}p_1^2a_{10}^3 - 2k_{12}p_2^2a_{10}a_{20}^2) \\
& + \tfrac{1}{4}(-3k_{13}a_{10}^2a_{20} + k_{12}p_1^2a_{10}^2a_{20}\cos\sigma_1 - 2k_{12}p_1p_2a_{10}^2a_{20})\cos\sigma_1 + f_{11}\cos\varphi_1, \\
R_{12} = {} & 2\omega_{20}\omega_{21}a_{20} + \mu_{12}p_2\omega_{21}a_{20} + \tfrac{1}{4}(-6k_{13}a_{10}^2a_{20} - 3k_{13}a_{20}^3 - 2k_{12}p_1^2a_{10}^2a_{20} - k_{12}p_2^2a_{20}^3) \\
& + \tfrac{1}{4}(-k_{13}a_{10}^3 + k_{12}p_1^2a_{10}^3)\cos\sigma_1, \\
R_{13} = {} & \mu_{11}\omega_{10}a_{10} - \tfrac{1}{4}(-3k_{13}a_{10}^2a_{20} + k_{12}p_1^2a_{10}^2a_{20} - 2k_{12}p_1p_2a_{10}^2a_{20})\sin\sigma_1 + f_{11}\sin\varphi_1, \\
R_{14} = {} & \mu_{11}\omega_{20}a_{20} + \tfrac{1}{4}(-k_{13}a_{10}^3 + k_{12}p_1^2a_{10}^3)\sin\sigma_1, \\
R_{21} = {} & 2p_1\omega_{10}\omega_{11}a_{10} + \mu_{21}\omega_{11}a_{10} + \tfrac{1}{4}(-k_{22}p_1a_{10}^3 - 2k_{22}p_1a_{10}a_{20}^2 - 3k_{23}p_1^3a_{10}^3 - 6k_{23}p_1p_2^2a_{10}a_{20}^2) \\
& + \tfrac{1}{4}(2k_{22}p_1a_{10}^2a_{20} - k_{22}p_2a_{10}^2a_{20} + 3k_{23}p_1^2p_2a_{10}^2a_{20})\cos\sigma_1, \\
R_{22} = {} & 2p_2\omega_{20}\omega_{21}a_{20} + \mu_{21}\omega_{21}a_{20} + \tfrac{1}{4}(-2k_{22}p_2a_{10}^2a_{20} - k_{22}p_2a_{20}^3 - 6k_{23}p_1^2p_2a_{10}^2a_{20} - 3k_{23}p_2^3a_{20}^3) \\
& + \tfrac{1}{4}(-k_{22}p_1a_{10}^3 + k_{23}p_1^3a_{10}^3)\cos\sigma_1, \\
R_{23} = {} & -\mu_{22}p_1a_{10}\omega_{10} + \tfrac{1}{4}(2k_{22}p_1a_{10}^2a_{20} - k_{22}p_2a_{10}^2a_{20} + 3k_{23}p_1^2p_2a_{10}^2a_{20})\sin\sigma_1, \\
R_{24} = {} & -\mu_{22}p_2a_{20}\omega_{20} - \tfrac{1}{4}(-k_{22}p_1a_{10}^3 + k_{23}p_1^3a_{10}^3)\sin\sigma_1 \quad \text{and} \quad \sigma_1 = \varphi_2 - 3\varphi_1.
\end{aligned}$$

It can be seen that R_{11}–R_{24} are functions of the frequencies (ω_{11}, ω_{21}), amplitudes (a_{10}, a_{20}) and phase angles (φ_1, φ_2). Let a particular solution set of q_{11} and q_{21} assume the following form:

$$q_{11} = P_{11}\cos(\tau_1+\varphi_1) + P_{12}\cos(\tau_2+\varphi_2) + P_{13}\sin(\tau_1+\varphi_1) + P_{14}\sin(\tau_2+\varphi_2), \tag{33}$$

$$q_{21} = P_{21}\sin(\tau_1+\varphi_1) + P_{22}\sin(\tau_2+\varphi_2) + P_{23}\cos(\tau_1+\varphi_1) + P_{24}\cos(\tau_2+\varphi_2). \tag{34}$$

By substituting the last two equations into Eqs. (31) and (32), equating the coefficients of $\cos(\tau_n+\varphi_n)$ and $\sin(\tau_n+\varphi_n)$ in each of the equations gives

$$\begin{bmatrix} k_{11}-\omega_{10}^2 & -\mu_{12}\omega_{10} \\ -\mu_{21}\omega_{10} & k_{21}-\omega_{10}^2 \end{bmatrix} \begin{Bmatrix} P_{11} \\ P_{21} \end{Bmatrix} = \begin{bmatrix} R_{11} \\ R_{21} \end{bmatrix}, \quad \begin{bmatrix} k_{11}-\omega_{20}^2 & -\mu_{12}\omega_{20} \\ -\mu_{21}\omega_{20} & k_{21}-\omega_{20}^2 \end{bmatrix} \begin{Bmatrix} P_{12} \\ P_{22} \end{Bmatrix} = \begin{bmatrix} R_{12} \\ R_{22} \end{bmatrix},$$

$$\begin{bmatrix} k_{11}-\omega_{10}^2 & \mu_{12}\omega_{10} \\ \mu_{21}\omega_{10} & k_{21}-\omega_{10}^2 \end{bmatrix} \begin{Bmatrix} P_{13} \\ P_{23} \end{Bmatrix} = \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix}, \quad \begin{bmatrix} (k_{11}-\omega_{20}^2) & \mu_{12}\omega_{20} \\ \mu_{21}\omega_{20} & (k_{21}-\omega_{20}^2) \end{bmatrix} \begin{Bmatrix} P_{14} \\ P_{24} \end{Bmatrix} = \begin{bmatrix} R_{14} \\ R_{24} \end{bmatrix}.$$

With reference to Eq. (15) on the natural frequency, one can write

$$\begin{vmatrix} k_{11}-\omega_{10}^2 & -\mu_{12}\omega_{10} \\ -\mu_{21}\omega_{10} & k_{21}-\omega_{10}^2 \end{vmatrix} = 0, \quad \begin{vmatrix} k_{11}-\omega_{20}^2 & -\mu_{12}\omega_{20} \\ -\mu_{21}\omega_{20} & k_{21}-\omega_{20}^2 \end{vmatrix} = 0.$$

Existence of the non-trivial solutions requires

$$\begin{vmatrix} k_{11}-\omega_{10}^2 & R_{11} \\ -\mu_{21}\omega_{10} & R_{21} \end{vmatrix}=0,\quad \begin{vmatrix} k_{11}-\omega_{20}^2 & R_{12} \\ -\mu_{21}\omega_{20} & R_{22} \end{vmatrix}=0,\quad \begin{vmatrix} k_{11}-\omega_{10}^2 & R_{13} \\ \mu_{21}\omega_{10} & R_{23} \end{vmatrix}=0,\quad \begin{vmatrix} (k_{11}-\omega_{20}^2) & R_{14} \\ \mu_{21}\omega_{20} & R_{24} \end{vmatrix}=0. \quad (35)$$

Eqs. (35) constitutes the solvability conditions which contain six unknown variables, namely, the two frequencies (ω_{11}, ω_{21}), the two amplitudes (a_{10}, a_{20}) and the two phase angles (φ_1, φ_2). One can choose one of the six unknowns as an independent variable and the remaining five variables expressed in terms of the independent variable from the above solvability conditions and Eq. (15).

It is worth pointing out that the entire MDLP procedure can be easily and efficiently conducted by Matlab. This level of convenience is not equally enjoyed by other perturbation methods such as the method of multiple scales and the Krylov–Bogoliubov–Mitropolsky method.

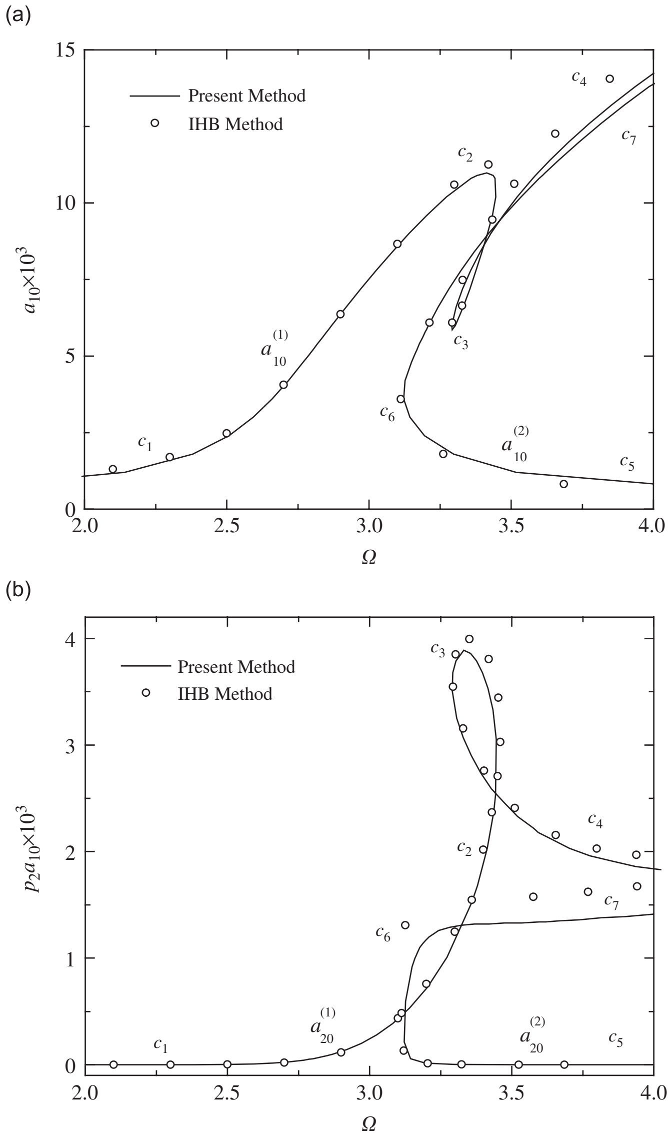

Fig. 2. Frequency response curves when Ω near ω_{10} with $f_{11}=0.0055$ and $\mu_{11}=\mu_{22}=0.04$.

Riedel and Tan [22] investigated the internal resonance response of an axially moving strip. Following their chosen system parameters which are typical for the belt drive system,

$$v_1^2 = 1124, \quad v_f^2 = 0.03 \quad \text{and} \quad v = 0.6$$

will be employed throughout this section. From the parameters, one can get

$$\omega_{10} = 2.82232, \quad \omega_{20} = 9.13980, \quad \mu_{12} = \mu_{21} = 3.2, \quad k_{11} = 9.23882, \quad k_{12} = 3372\pi^4, \quad k_{13} = 421.5\pi^4,$$
$$k_{21} = 72.0226, \quad k_{22} = 3372\pi^4, \quad k_{23} = 6744\pi^4, \quad p_1 = 0.14099 \quad \text{and} \quad p_2 = -2.5403.$$

With them, the second natural frequency of the system is nearly three times of the first natural frequency and therefore internal resonance will occur.

Fig. 2 shows the frequency response curves of the systems with $f_{11} = 0.0055$, $\mu_{11} = \mu_{22} = 0.04$ and $\Omega \approx \omega_{10}$. Fig. 2(a) shows the $\Omega - a_{10}$ curve and Fig. 2(b) shows the $\Omega - p_2 a_{20}$ curve where a_{10} and $p_2 a_{20}$ defined in Eq. (29) are the amplitudes of the first harmonic terms of the first variable q_1 and the third harmonic terms of the second variable q_2, respectively. In the figures, the solid lines and small circles represent the results of the

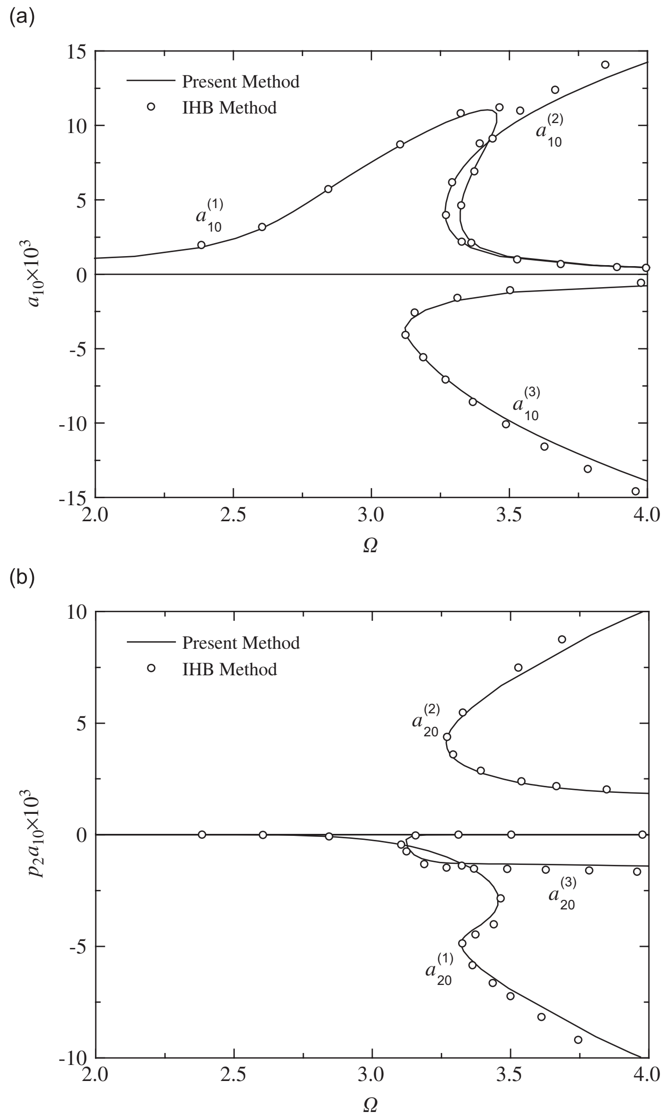

Fig. 3. Frequency response curves when Ω near ω_{10} with $f_{11} = 0.0055$ and $\mu_{11} = \mu_{22} = 0$.

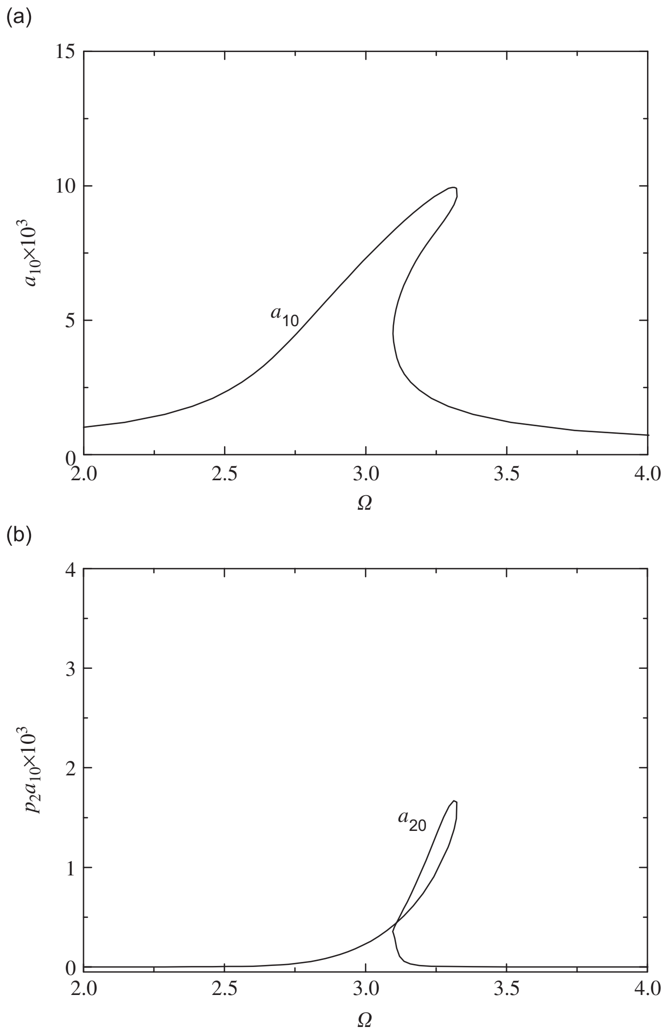

Fig. 4. Frequency response curves when Ω near ω_{10} with $f_{11} = 0.0055$ and $\mu_{11} = \mu_{22} = 0.15$.

MDLP and IHB methods, respectively. Internal resonance can be noted from the amplitudes of the responding modes.

Both a_{10} and a_{20} have two solutions. The first solutions $a_{10}^{(1)}$ and $p_2a_{20}^{(1)}$ exhibit the characters of internal resonance, which is similar to those of clamped-hinged beams computed with an alternative perturbation procedure of multiple scales by Chen et al. [28], thin plates computed with IHB method by Lau et al. [29] and frames computed with the combination of the IHB method and the finite element method by Leung and Fung [30], respectively. These similarities are due to the common cubic nonlinearity and frequency distribution. However, for the second solutions $a_{10}^{(2)}$ and $p_2a_{20}^{(2)}$, there is no exchange of responding modes. Internal resonance occurs in the first solution only.

Fig. 3 shows the frequency response curves of the system for $f_{11} = 0.0055$ and zero external damping, i.e., $\mu_{11} = \mu_{22} = 0$. One can note that the response curves also have loops which demonstrate the characteristic of internal resonance. Fig. 4 shows the frequency response curves of the system for $f_{11} = 0.0055$ and $\mu_{11} = \mu_{22} = 0.1$. There is no loop in the response curves. The external damping coefficients are sufficiently large to annihilate any internal resonance.

It can be seen from Figs. 2–4 that the results of the MDLP method are almost identical to that obtained by the IHB method for the first solutions $a_{10}^{(1)}$ and $p_2a_{20}^{(1)}$. For the second solutions $a_{10}^{(2)}$ and $p_2a_{20}^{(2)}$, the result obtained by IHB method and MDLP method are almost identical from the points c_5's to turning points c_6's.

After c_6's, the results of two methods are only slightly different. This leads to the conclusion that the MDLP method is a reliable analytic method for periodic solutions of multiple degree of freedom systems.

Comparing with the IHB method, the MDLP method can yield the solution more readily. In the IHB method, it is difficult to choose the initial values for iteration. However, the IHB method can give more exact solution especially for strongly nonlinear systems. Therefore, these two methods can complement each other. The results of MDLP can be taken as the initial values of iteration in the IHB method such that one can obtain the solution more easily and exactly.

Same as the afore-discussed fundamental resonance, other resonances such as sub-harmonic resonance, super-harmonic resonance and combination resonance can also be studied by using the MDLP method.

4. Concluding remarks

The MDLP method is extended to analyze the nonlinear vibration of axially moving systems. The considered typical example has illustrated that the MDLP method is more straightforward and efficient than other perturbation methods for multiple degree-of-freedom systems such as the method of multiple scales and the Krylov–Bogoliubov–Mitropolsky method.

The forced responses of an axially moving beam with the excitation frequency Ω near the first natural frequency ω_{10} were investigated. When the damping is small, all the response curves exhibit the same internal resonance characteristics as that of non-transferring thin plates and beams because all these systems possess cubic nonlinearity and similar frequency distribution.

When the vibration amplitude is small, the predictions of the MDLP method are in good agreement with those of the IHB method. Two methods can complement each other.

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